Lecture 6 (1/14/22).

A tangent (not in Conway):
<u>Hyperbolic (Poincaré) metric in $\mathbb{D}$.</u>

Previously, we have discussed two geometric models. In terms of metric spaces:

- $X = \mathbb{C}$, $d_E(z,w) = |z-w|$. (Euclidean metric)
  - Complete, not compact.
  - Invariant under translations + rotations.
  - In terms of Riemannian geometry:
    - * Geodesics (shortest paths) are straight lines.
    - * Constant curvature $= 0$

- $X = \mathbb{C}_\infty$ (realized at Riemann sphere $S^2 \subseteq \mathbb{R}^3$), $d_{F.S.}(z,w) = \dfrac{2|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}}$

(Fubini-Study metric).

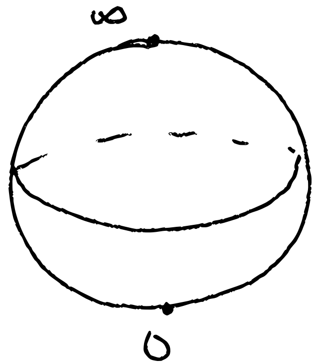
- Compact ($\Rightarrow$ complete)
- Invariant under Möbius $T = \dfrac{az+b}{cz+d}$ s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PU(2,\mathbb{C})$

- In Riemannian geometry:
  * Geodesics are circles on $\mathbb{C}_\infty$ (incl. lines).
  * Constant curvature > 0.

Invariance of d w.r.t. Möbius $T \Rightarrow$

$$d_{FS}(z,w) = d_{FS}\left(0, \frac{w-z}{1+\bar{z}w}\right)$$

rotation of $S^2 \subseteq \mathbb{R}^3$.



Hyperbolic disk.

There is a third basic geometry w/ $\mathbb{D} = \{|z|<1\}$ as the underlying space and the Poincaré/hyperbolic metric $d_H(z,w)$, which can be described as follows.

We first define a function $r: \mathbb{D}\times\mathbb{D} \to [0,\infty)$ by

$$r(z,w) = d_E(0, \varphi_z(w)) = |\varphi_z(w)|,$$

where $\varphi_a(z) = \frac{z-a}{1-\bar{a}z} \in \mathrm{Aut}(\mathbb{D})$.

Function r has properties:

- $r(z,w) \geq 0$ w/ "=" iff $z=w$
- $r(z,w) = r(w,z)$

This almost makes r a metric (Δ-ineq. is missing), but we also want d_H to be <u>complete</u>, which one can show is $\Leftrightarrow$ $d(0,w) \to \infty$ as $w \to \partial\mathbb{D}$.
($r < \underline{1}$ on $\mathbb{D}\times\mathbb{D}$),

Def. ① $d_H(z,w) = \log\frac{1+r}{1-r}$, $r = r(z,w)$.
is the Poincaré/hyperbolic metric (distance).

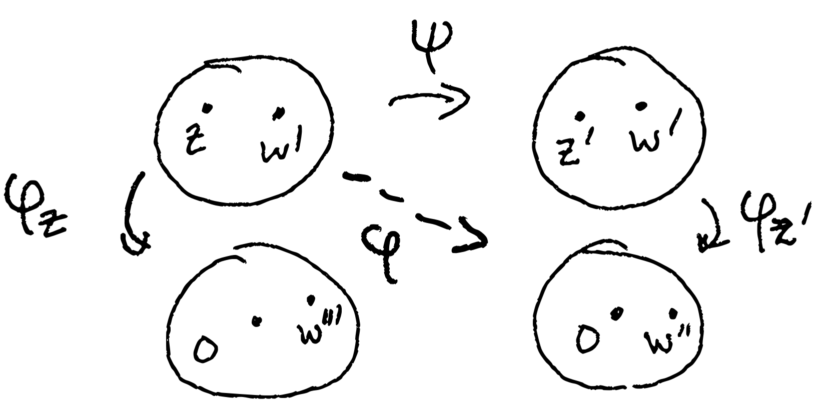
Thm 1. d_H is a complete metric, invariant under $\mathrm{Aut}(\mathbb{D})$.

__Sketch of Pf.__ To show d_H metric, we need to verify Δ-ineq. Let's save this.

- __Complete.__ $d_H(0,w) = \log \frac{1+|w|}{1-|w|} \to \infty$

  as $w \to \partial\mathbb{D}$.

- __Invariance under $\mathrm{Aut}(\mathbb{D})$.__ By previous thm, $\varphi \in \mathrm{Aut}(\mathbb{D}) \Leftrightarrow \varphi = e^{i\theta}\varphi_a$ for some $a \in \mathbb{D}$. We note we can replace φ_z in def. of $d_H(z,\cdot)$ by any $\varphi \in \mathrm{Aut}(\mathbb{D})$ s.t. $\varphi(z)=0$. Invariance follows from:



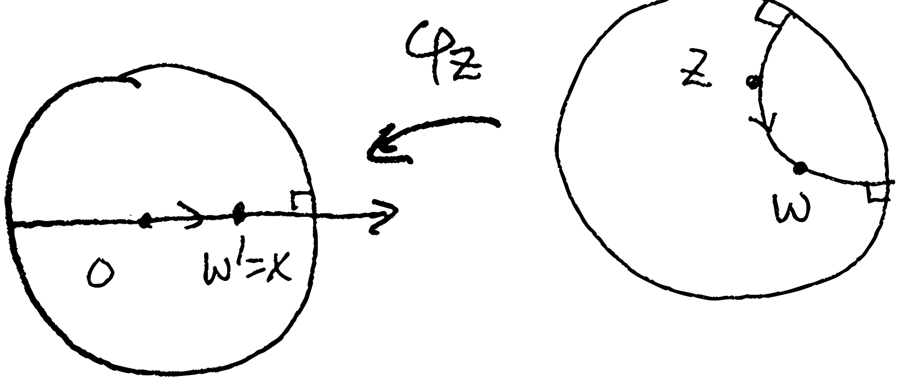
• $\underline{\Delta\text{-ineq}}$. One notes that for $x \in (0,1)$

$$d_H(0,x) = \log \frac{1+x}{1-x} = \int_0^x \frac{dr}{1-r^2}.$$

One shows

$$d_H(z,w) = \int_{\gamma_{z,w}} \frac{d\zeta}{1-|\zeta|^2} \quad \text{where}$$

$\gamma_{z,w}$ is circular arc:



and

$$d_H(z,w) = \inf_{\substack{\gamma \subseteq \mathbb{D} \text{ s.t.} \\ \gamma(0)=z,\, \gamma(1)=w}} \int_\gamma \frac{d\zeta}{1-|\zeta|^2},$$

Riemannian distance induced by $ds^2 = \dfrac{|dz|^2}{(1-|z|^2)^2}$.

Δ-ineq. follows easily (by being induced by Riemannian metric). $\boxed{\square}$

In this context.

<u>Schwarz Lemma</u>. Let f be analytic in $\mathbb{D}$, $f: \mathbb{D} \to \mathbb{D}$. Then $d_H(f(a), f(b)) \le d_H(a,b)$ w/ "=", $a \ne b$, iff $f \in \mathrm{Aut}(\mathbb{D})$.

<u>Pf.</u> Let $a' = f(a)$, $b' = f(b)$. Consider $g = \varphi_{a'} \circ f \circ \varphi_{-a}$. Then, $|g| \le 1$ and

$$g(0) = \varphi_{a'}(f(\varphi_{-a}(0))) = \varphi_{a'}(f(a)) = \varphi_{a'}(a') = 0.$$

$z = \varphi_a(w)$

By S.L., $|g(z)| \le |z| \Rightarrow |(\varphi_{a'} \circ f)(w)| \le |\varphi_a(w)|$ or w/ $w = b \Rightarrow b' = f(b)$

$$|\varphi_{a'}(b')| \le |\varphi_a(b)|$$

Thus, $r(a',b') \le r(a,b) \Rightarrow$

$d_H(a',b') \le d_H(a,b)$.

Equality $\Leftrightarrow$ $|g(z)| = |z|$ w/ $z = \varphi_a(b) \neq 0$

By S.L again, $g(z) = cz$. $\Rightarrow$

$f = \varphi_{-a'} \circ c\varphi_a \in \mathrm{Aut}(\mathbb{D})$. $\boxtimes$